Partial Differential Equations - Resit Exam

You have 3 hours to complete this exam. Please show all work. Each question has point values listed next to it for a total of 90 points. (10 points are free for a total of 100 points) This exam has two sides to it. Be sure to quote clearly any theorems you use from the textbook or class. Good luck!

1. (20 points) Prove that if $u(x,y)$ is harmonic in a bounded region Ω and u is $C^1(\overline{\Omega})$ then $w = |\nabla u|^2$ attains its maximum on $\partial\Omega$, the boundary of Ω. (Hint, what is the sign of Δw?)

2. (15 points) For each of the following equations find the regions in the plane xy where they are elliptic, parabolic or hyperbolic, and determine the corresponding characteristic curves.

$$\begin{aligned} a)&\quad 2u_{xx} - 4u_{xy} - 6u_{yy} + u_x = 0 \quad (x,y) \in \mathbb{R}^2 \\ b)&\quad 4u_{xx} + 12u_{xy} + 9u_{yy} - 2u_x + u = 0 \quad (x,y) \in \mathbb{R}^2 \\ c)&\quad u_{xx} - x^2 y u_{yy} = 0 \quad (x,y) \in \mathbb{R} \times \mathbb{R}_+ \end{aligned}$$

3. (20 points) Show that the solution to the one dimensional heat equation in free space

$$\begin{aligned} &\partial_t u(t,x) = \partial_x^2 u(t,x) \quad [0,T] \times \mathbb{R} \\ &u(0,x) = f(x) \end{aligned}$$

for $f(x) \in C^2(\mathbb{R})$ is

$$u(t,x) = K \star f = \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi t}} e^{-\frac{|x-y|^2}{4t}} f(y)\, dy$$

Prove that $\lim_{t \to 0} K \star f = f(x)$ in your derivation.

4. (30 points) Find the possible separated solutions $\phi(x,t) = X(x)T(t)$ to the equation

$$\partial_x^2 \phi = \partial_t^2 \phi + \partial_t \phi$$

If $\phi(0,t) = \phi(1,t) = 0, \partial_t(x,0) = 0$, and $\phi(x,0) = \sin^2(\pi x)$ what is $\phi(x,t)$? You may use the fact that the Fourier series coefficients of $\sin^2(\pi x)$ are

$$\frac{4}{n\pi}\left(\frac{(-1)^n - 1}{n^2 - 4}\right)$$

without proof.

5. (5 points) Suppose that a function w satisfies the advection-diffusion equation $w_t + 2w_x = w_{xx}$ for $0 < x < 1$ and $t > 0$ together with Robin boundary conditions $w_x = 2w$ at $x = 0$ and $x = 1$ and the initial condition $w(x, 0) = 6x$ for $0 < x < 1$. Show that the *total mass* defined by

$$M(t) = \int_0^1 w(x,t)\,dx$$

satisfies $dM(t)/dt = 0$ and deduce that $M(t) = 3$ for all $t \geq 0$.

MoL